# Radiation conditions in Finite Elements Method for finite inhomogeneous structures.

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To begin our work, we are interested in a 2D problem but with full acoustic and electronic fields. The configuration is depicted in the figure 1. The frontier $\Gamma$ divide space into two parts, the homogeneous one and the inhomogeneous part ($\Omega$). The last one contains a finite network of transducers which is embedded in a surrounding medium and lies on a backing. The inhomogeneous region describes an area varying between $x_{11}$ and $x_{12}$ on $x_1$-direction and $x_{21}$ and $x_{22}$ on $x_2$-direction. All fields are calculated by dynamic piezo-electric finite elements method with a displacement-potential formulation. We suppose Dirichlet boundary conditions for potential, i.e., the potential is specified. The admittance is deduced from the charges firstly calculated.

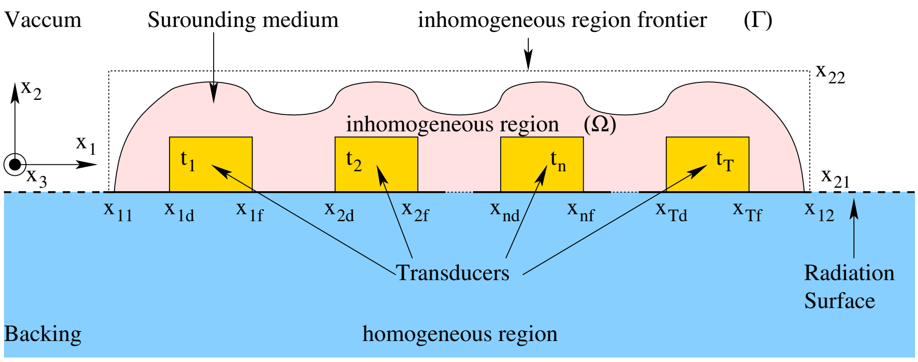


Figure 1: Sample of studied configurations: A finite network of acoustic transducers lied on a backing and embedded in surrounding medium. The $\omega$ region is demarcated by the light dashed-line. This line is called $\Gamma$. The $\Omega$ region describes an inhomogeneous area.

To simplify the problem, we only apply radiation boundary conditions on the $x_1$-direction which becomes the new $\Gamma$. So, $\Gamma$ is depicted by thick black line under the transducer in figure 1. this line coincides with $x_2 = 0$.

From the principle of Hamilton and the variational approach, the stability condition on $\Gamma$ is defined here for our configuration[1] :

$$\iiint_{\Omega} \left( -\rho \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial T_{ij}}{\partial x_j} \right) \delta u_i d\Omega = \iint_{\Gamma} T_{ij} n_j \delta u_i d\Gamma, \tag{1}$$

where $\rho$ is the density, $u_i$ the displacement for the $x_i$-direction for $i \in [1, 3]$ and $u_4$ is the electric potential, $T$ the stress tensor with $T_{4i}$ the $i$ component of the electric displacement vector, $\delta u_i$ the virtual increasing of the displacement (or potential for $i = 4$) for the $x_i$-direction and $n_j$ is the vector normal to the $x_j$-direction.

Now we write the radiation conditions on $\Gamma$:

$$T_{ij}n_j = \int_{-\infty}^{+\infty} g_{ijk}(x_1 - x'_1)n_j u_k(x'_1)dx'_1, \tag{2}$$

where $g(x_1)$ is the matrix of the green functions relating the stress tensor $T$ to the displacements $u$. This relation allows us to determine the left part of the equation (1)

Problem! The green functions cannot be easily defined. Indeed, It has been shown the asymptotic behavior of the spectral Green function is proportional to the slowness $s_1$[2]. Whereas this properties does not disturb the calculation of the Green function for periodic structures[3], the non-periodic case cannot be calculate by this way and leads to a non-periodic Green function does not converge ($\infty$).

To avoid this trouble, we rewrite the boundary condition as a function of the stress tensor[4]:

$$u_i(x_1) = \int_{-\infty}^{+\infty} G_{ijk}(x_1 - x'_1)T_{jk}(x'_1)n_k dx'_1, \tag{3}$$

where $G(x_1)$ is the matrix of the green functions relating the displacements $u$ to the strain tensor $T$. In the chosen coordinates system, the normal to $\Gamma$ is $n_2$. So, equation (3) can be rewritten as following,

$$u_i(x_1) = \int_{-\infty}^{+\infty} G_{ij2}(x_1 - x'_1)T_{j2}(x'_1)dx'_1. \tag{4}$$

We can calculate all its elements from its Fourier transform :

$$G_{ijk} = \int_{-\infty}^{+\infty} \hat{G}_{ijk} \exp(-jk_1x_1)dk_1 \tag{5}$$

where $\hat{G}$ is the Fourier transform of $G$ and $k_1$ is the wave number for the $x_1$-direction. The wave number is a function of the pulsation $\omega$ and the slowness $s_1$ for the $x_1$-direction, $k_1 = \omega s_1$. thus the differential $dk_1$ becomes $dk_1 = \omega ds_1$. If we consider the canonical concept the Fourier transform can be write as :

$$\hat{G}_{ijk}(\omega, s_1) = \frac{\hat{H}_{ijk}(s_1)}{\omega} \tag{6}$$

where $\hat{H}$ is the part of the function related to $\hat{G}$ which do not depend on $\omega$. Thus, the green function do not depend on the frequency. So, the equation (5) becomes:

$$G_{ijk}(\omega x_1) = \int_{-\infty}^{+\infty} \hat{H}_{ijk}(s_1) \exp(-j\omega x_1 s_1)ds_1 \tag{7}$$

The variational form of the equation (4) to calculate the stress tensor of the equation (1):

$$\int_{x_1=-\infty}^{x_1=+\infty} \delta u_i u_i(x_1)dx_1 = \int_{x_1=-\infty}^{x_1=+\infty} \delta u_i \int_{x'_1=-\infty}^{x'_1=+\infty} G_{ij2}(x_1 - x'_1)T_{j2}(x'_1)dx'_1 dx_1. \tag{8}$$

We substitute $G_{ij2}$ in (8) by its form obtained in (7):

$$\int_{x_1=-\infty}^{x_1=+\infty} \delta u_i u_i(x_1) dx_1 = \int_{x_1=-\infty}^{x_1=+\infty} \delta u_i \int_{x_1'=-\infty}^{x_1'=+\infty} \int_{-\infty}^{+\infty} \hat{H}_{ij2}(s_1) \exp(-j\omega(x_1 - x_1')s_1) ds_1 T_{j2}(x_1') dx_1' dx_1. \quad (9)$$

After some algebraic calculations, once can write the canonical spectral Green tensor $\hat{H}_{ij2}(s_1)$ for an isotropic medium :

$$\hat{H}_{k=2}(s_1) = \frac{j}{C_{66}\Delta} \begin{pmatrix} j\frac{s_t^2\chi_2}{s_1^3} & \frac{2\chi_1\chi_2-(1+\chi_2^2)}{s_1} & 0 \\ -\frac{2\chi_1\chi_2-(1+\chi_2^2)}{s_1} & j\frac{s_t^2\chi_1}{s_1^3} & 0 \\ 0 & 0 & \frac{\Delta}{\chi_2 s_1} \end{pmatrix} \quad (10)$$

where $\chi_1 = \sqrt{1 - \frac{s_l^2}{s_1^2}}$ and $\chi_2 = \sqrt{1 - \frac{s_t^2}{s_1^2}}$. Let $s_t = \sqrt{\frac{\rho}{C_{66}}}$ and $s_l = \sqrt{\frac{\rho}{C_{11}}}$ with $\rho$ the density and $C_{66}$ and $C_{11}$ two reduced components of the rigidity tensor for an isotropic material. Thus, $\Delta$ is defined below:

$$\Delta = (1 + \chi_2^2)^2 - 4\chi_1\chi_2 \quad (11)$$

This spectral green function is holomorphe. In other words, this function is defined and continue on the whole domain of integration. However, we must split the Green function in several parts in order to improve the Fourier integration time. We distinguish three parts: the asymptotic behavior when $s_1$ tends to zero, all the different poles. First, its asymptotic behavior $\hat{H}_{k=2}^{\infty}(s_1)$ tends to 0 when $s_1$ tends to $+\infty$ where :

$$\hat{H}_{k=2}^{\infty}(s_1) = -\frac{j}{\rho(1 + \frac{C_{12}}{C_{11}})s_1} \begin{pmatrix} js_t^2 & -s_l^2 & 0 \\ s_l^2 & js_t^2 & 0 \\ 0 & 0 & -s_t^2(1 + \frac{C_{12}}{C_{11}}) \end{pmatrix} \quad (12)$$

Last, there is only one pole for this function which is defined by the solution of the equation $\Delta = 0$. The slowness solution of this equation correspond to the Rayleigh wave[5]. Indeed, we can write this equation in an other way :

$$R^3 - 8(R-1)(R - 1 - \frac{C_{12}}{C_{11}}) = 0 \quad (13)$$

where $R = \frac{st^2}{s_1^2}$. This form was established by Lord Rayleigh in 1885. Moreover, there is only one positive solution $s_1^R$ which can be approximated by the Viktorov formula. In this way, we are able to calculate the Green function for $s_1 = s_1^R$, $\hat{H}_{k=2}^R = \hat{H}_{k=2}(s_1^R) = \delta(s_1^R)$ (where $\delta$ is Dirac distribution). Thus the canonical Green function can be written as a summation of three terms :

$$\hat{H}_{k=2}(s_1) = \hat{H}_{k=2}^{(0)}(s_1) + \hat{H}_{k=2}^{\infty}(s_1) + \hat{H}_{k=2}^R, \quad (14)$$

where $\hat{H}_{k=2}^{(0)}(s_1)$ is the Green function without the asymptotic and polar contributions. For the asymptotical and polar contribution, we can easily calculate the inverse Fourier transform :

$$H_{k=2}^R(\omega X) = A\frac{\exp j\omega s_1^R X}{2\pi} \quad (15)$$

$$H^{\infty}_{k=2}(\omega X) = j\frac{B}{2}(1 - 2H_e(-\omega X)) \tag{16}$$

In equation (15) and (16), $H^{R}_{k=2}$ and $H^{\infty}_{k=2}$ are respectively the inverse Fourier transforms of the polar and asymptotical contribution of the spectral Green function, with $A$ and $B$ two matrices which depends on their coefficients. The function $H_e$ is the Heaviside function. However, we still have to calculate the inverse Fourier transform of $\hat{H}^{(0)}_{k=2}$, i.e., $H^{(0)}_{k=2}$. Thus, we should obtain the total inverse Fourier transform, $H_{k=2}$.

Thus, we can rewrite equation (9) with the new notation :

$$\begin{aligned}\int_{x_1=-\infty}^{x_1=+\infty} \delta u_i u_i(x_1)dx_1 = \int_{x_1=-\infty}^{x_1=+\infty} \delta u_i \int_{x'_1=-\infty}^{x'_1=+\infty} \int_{-\infty}^{+\infty} \Big(\hat{H}^{(0)}_{ij2}(s_1) + \cdots \\ \cdots + \hat{H}^{\infty}_{ij2}(s_1) + \hat{H}^{R}_{ij2}\Big) \exp(-j\omega(x_1 - x'_1)s_1)ds_1 T_{j2}(x'_1)dx'_1 dx_1.\end{aligned} \tag{17}$$

Of course we can develop the same method to find the spectral Green tensor for an other symmetry.

Applying the discretization process, equation (17) becomes :

$$\begin{aligned}\int_\Gamma \delta u_i u_i(x_1)dx_1 = \int_{-\infty}^{+\infty} (\hat{H}^{(0)}_{ij2}(s_1) + \hat{H}^{\infty}_{ij2}(s_1) + \hat{H}^{R}_{ij2})\times \\ \left(\sum_{e=1}^{N_e}\sum_{m=1}^{\eta_e} \delta u_i{}^{(em)} \int_{\Gamma_e} P^{em}_i(x^e_1)\exp(-j\omega x^e_1 s_1)d\Gamma_e \times \right. \\ \left. \sum_{\epsilon=1}^{\epsilon=N_e}\sum_{\mu=1}^{\eta_\epsilon} T^{t(\epsilon\mu)}_{j2} \int_{\Gamma_\epsilon} Q^{(\epsilon\mu)}_j(x^\epsilon_1)\exp(j\omega x^\epsilon_1 s_1)d\Gamma_\epsilon\right) ds_1,\end{aligned} \tag{18}$$

where $N_e$ is the number of elements, $\eta_e$ is the number of nodes for element $e$ and $P^{em}_i$ is the interpolation polynomial for $m^{th}$ node of element $e$ (similarly for $Q^{\epsilon\mu}_j$). This way does not lead to an definite form of the Fourier transforms. So, we must rewrite (18) to take into account first the calculation of the inverse Fourier transforms of the Green tensor and secondly the convolution product.

$$\begin{aligned}\int_\Gamma \delta u_i u_i(x_1)dx_1 = \sum_{e=1}^{N_e}\sum_{m=1}^{\eta_e} \delta u_i{}^{(em)} \int_{\Gamma_e} P^{(em)}_i(x^e_1) \\ \left(\sum_{\epsilon=1}^{\epsilon=N_e}\sum_{\mu=1}^{\eta_\epsilon} T^{(\epsilon\mu)}_{j2} \int_{\Gamma_\epsilon} Q^{(\epsilon\mu)}_j(x^\epsilon_1) \left(\int_{-\infty}^{+\infty} (\hat{H}^{(0)}_{ij2}(s_1) + \cdots\right.\right. \\ \left.\left. \cdots + \hat{H}^{\infty}_{ij2}(s_1) + \hat{H}^{R}_{ij2}) \exp(-j\omega(x^e_1 - x^\epsilon_1)s_1)ds_1\right) d\Gamma_\epsilon\right) d\Gamma_e,\end{aligned} \tag{19}$$

if we replace the Green functions in equation (19) by their forms obtained in equations (15) and (16), equation (19) becomes:

$$
\begin{aligned}
\int_\Gamma \delta u_i u_i(x_1)dx_1 = &\sum_{e=1}^{N_e}\sum_{m=1}^{\eta_e} \delta u_i{}^{(em)} \int_{\Gamma_e} P_i^{(em)}(x_1^e) \\
&\sum_{\epsilon=1}^{N_\epsilon}\sum_{\mu=1}^{\eta_\epsilon} \int_{\Gamma_\epsilon} Q_j^{(\epsilon\mu)}(x_1^\epsilon)\left(H_{ij2}^{(0)}(\omega(x_1^e - x_1^\epsilon)) + \cdots\right. \\
&\left.\cdots + H_{ij2}^{\infty}(\omega(x_1^e - x_1^\epsilon)) + H_{ij2}^{R}(\omega(x_1^e - x_1^\epsilon))\right) d\Gamma_\epsilon d\Gamma_e T_{j2}^{(\epsilon\mu)}.
\end{aligned}
\tag{20}
$$

If we put $G_{ij}^{(e\epsilon m\mu)} = \int_{\Gamma_e}\int_{\Gamma_\epsilon} P_i^{(em)}(x_1^e) Q_j^{(\epsilon\mu)}(x_1^\epsilon) H_{ij2}(\omega(x_1^e - x_1^\epsilon)) d\Gamma_\epsilon d\Gamma_e$, we can rewrite (20) in a simplier manner:

$$
\int_\Gamma \delta u_i u_i(x_1)dx_1 = \sum_{e=1}^{N_e}\sum_{m=1}^{\eta_e}\sum_{\epsilon=1}^{N_\epsilon}\sum_{\mu=1}^{\eta_\epsilon} \delta u_i{}^{(em)} G_{ij}^{(e\epsilon m\mu)} T_{j2}^{(\epsilon\mu)}. \tag{21}
$$

The left side of equation (21) leads to the same discretized form. Let $\Psi_{ij}^{(e\epsilon m\mu)} = \int_{\Gamma_e} K_i^{(em)} L_j^{(\epsilon\mu)} \delta_{ij}^{e\epsilon} d\Gamma_e$ where $K_i^{(em)}$ and $L_j^{(\epsilon\mu)}$ are respectively the polynomials related to $\delta u_i{}^{(em)}$ and $u_j^{(\epsilon\mu)}$. So, equation (21) becomes:

$$
\sum_{e=1}^{N_e}\sum_{\epsilon=1}^{N_\epsilon}\sum_{m=1}^{\eta_e}\sum_{\mu=1}^{\eta_\epsilon} \delta u_i{}^{(em)} \Psi_{ij}^{(e\epsilon m\mu)} u_j^{(\epsilon\mu)} = \sum_{e=1}^{N_e}\sum_{\epsilon=1}^{N_\epsilon}\sum_{m=1}^{\eta_e}\sum_{\mu=1}^{\eta_\epsilon} \delta u_i{}^{(em)} G_{ij}^{(e\epsilon m\mu)} T_{j2}^{t(\epsilon\mu)}. \tag{22}
$$

By a simple way, equation (22) can be put in a matrix form:

$$
< \delta u_i > (\Psi_{ij})\{u_j\} =< \delta u_i > (G_{ij})\{T_{j2}\}, \tag{23}
$$

The finite element scheme defines the interaction between all the nodes of the whole mesh. So, we must take of the redundancies in equation (22). For instance in our case, the $\eta_e^{th}$ node of the $e^{th}$ is equivalent to the first one of the $(e+1)^{th}$ element. So, all the interaction of these nodes must be summed. Thus, the dimensions of matrices $(\Psi_{ij})$ and $(G_{ij})$ are $((\sum_{e=1}^{N_e} N_{de}) - N_e + 1, (\sum_{e=1}^{N_e} N_{de}) - N_e + 1)$, ie the total number of nodes in the whole mesh.

The previous matrices can be filled from the sum of the matrices for each couple of element $e$ and $\epsilon$ (These matrices have the same dimensions than the total matrices $\Psi_{ij}$ and $G_{ij}$):

$$
\Psi_{ij}^{e\epsilon} = \delta_{ij}^{e\epsilon}
\begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \Psi_{ij}^{(e\epsilon 11)} & \cdots & \Psi_{ij}^{(e\epsilon 1\eta_\epsilon)} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \Psi_{ij}^{(e\epsilon\eta_e 1)} & \cdots & \Psi_{ij}^{(e\epsilon\eta_e\eta_\epsilon)} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}, \tag{24}
$$

and

$$G_{ij}^{e\epsilon} = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & G_{ij}^{(e\epsilon 11)} & \cdots & G_{ij}^{(e\epsilon 1\eta_\epsilon)} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & G_{ij}^{(e\epsilon\eta_e 1)} & \cdots & G_{ij}^{(e\epsilon\eta_e\eta_\epsilon)} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{25}$$

The sum of the sub matrices leads to the total matrices, $\Psi_{ij} = \sum_{e=1}^{N_e} \sum_{\epsilon=1}^{N_\epsilon} \Psi_{ij}^{e\epsilon}$ and $G_{ij} = \sum_{e=1}^{N_e} \sum_{\epsilon=1}^{N_\epsilon} G_{ij}^{e\epsilon}$. In equation (26), we note $A = \psi_{ij}^{((N_e-1)(N_e-1)\eta_{Ne-1}\eta_{Ne-1})}$, $B = \psi_{ij}^{((N_e-1)(N_e-1)(\eta_{Ne-1}-1)\eta_{Ne-1})}$, $C = \psi_{ij}^{((N_e-1)(N_e-1)(\eta_{Ne-1}-1)(\eta_{Ne-1}-1))}$ and $D = \psi_{ij}^{((N_e-1)(N_e-1)\eta_{Ne-1}(\eta_{Ne-1}-1))}$.

$$
\Psi_{ij} = \left\{ \begin{array}{cccccccccc}
\psi_{ij}^{(1111)} & \cdots & \psi_{ij}^{(111(\eta_1-1))} & \psi_{ij}^{(111\eta_1)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\psi_{ij}^{(11(\eta_1-1)1)} & \cdots & \psi_{ij}^{(11(\eta_1-1)(\eta_1-1))} & \psi_{ij}^{(11(\eta_1-1)\eta_1)} & 0 & \cdots & \cdots & \cdots & \vdots & \cdots \\
\psi_{ij}^{(11\eta_1 1)} & \cdots & \psi_{ij}^{(11\eta_1(\eta_1-1))} & \psi_{ij}^{(11\eta_1\eta_1)} + \psi_{ij}^{(2211)} & \psi_{ij}^{(2212)} & \cdots & \psi_{ij}^{(221(\eta_2-1))} & \psi_{ij}^{(221\eta_2)} & 0 & \cdots \\
0 & \cdots & 0 & \psi_{ij}^{(2221)} & \psi_{ij}^{(2222)} & \cdots & \psi_{ij}^{(222(\eta_2-1))} & \psi_{ij}^{(222\eta_2)} & \vdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots \\
\vdots & \vdots & \vdots & \psi_{ij}^{(22(\eta_2-1)1)} & \psi_{ij}^{(22(\eta_2-1)2)} & \cdots & \psi_{ij}^{(22(\eta_2-1)(\eta_2-1))} & \psi_{ij}^{(22(\eta_2-1)\eta_2)} & 0 & \cdots \\
\vdots & \vdots & \vdots & \psi_{ij}^{(22\eta_2 1)} & \psi_{ij}^{(22\eta_2 2)} & \cdots & \psi_{ij}^{(22\eta_2(\eta_2-1))} & \psi_{ij}^{(22\eta_2\eta_2)} + \psi_{ij}^{(3311)} & \psi_{ij}^{(3312)} & \cdots \\
\vdots & \vdots & \vdots & 0 & \cdots & \cdots & 0 & \psi_{ij}^{(3321)} & \psi_{ij}^{(3322)} & \ddots \\
\vdots & \cdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \ddots & \ddots
\end{array} \right.
$$

$$
\left. \begin{array}{cccccc}
\ddots & \ddots & \vdots & \vdots & \cdots & \vdots \\
\ddots & C & B & 0 & \cdots & 0 \\
\cdots & D & \psi_{ij}^{(N_e N_e 11)} + A & \psi_{ij}^{(N_e N_e 12)} & \cdots & \psi_{ij}^{(N_e N_e 1\eta_{Ne})} \\
\cdots & 0 & \psi_{ij}^{(N_e N_e 21)} & \psi_{ij}^{(N_e N_e 22)} & \cdots & \psi_{ij}^{(N_e N_e 2\eta_{Ne})} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
0 & 0 & \psi_{ij}^{(N_e N_e \eta_{Ne} 1)} & \psi_{ij}^{(N_e N_e \eta_{Ne} 2)} & \cdots & \psi_{ij}^{(N_e N_e \eta_{Ne}\eta_{Ne})}
\end{array} \right\} \tag{26}
$$

$$
G_{ij} = \left\{ \begin{array}{cccccc}
G_{ij}^{(1111)} & \cdots & G_{ij}^{(111(\eta_1-1))} & G_{ij}^{(111\eta_1)} + G_{ij}^{(1211)} & G_{ij}^{(1212)} & \cdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \cdots \\
G_{ij}^{(11(\eta_1-1)1)} & \cdots & G_{ij}^{(11(\eta_1-1)(\eta_1-1))} & G_{ij}^{(11(\eta_1-1)\eta_1)} + G_{ij}^{(12(\eta_1-1)1)} & G_{ij}^{(12(\eta_1-1)2)} & \cdots \\
G_{ij}^{(11\eta_1 1)} + G_{ij}^{(2111)} & \cdots & G_{ij}^{(11\eta_1(\eta_1-1))} + G_{ij}^{(211(\eta_1-1))} & G_{ij}^{(11\eta_1\eta_1)} + G_{ij}^{(12\eta_1 1)} + G_{ij}^{(211\eta_1)} + G_{ij}^{(2211)} & G_{ij}^{(2212)} + G_{ij}^{(12\eta_1 2)} & \cdots \\
G_{ij}^{(2111)} & \cdots & G_{ij}^{(211(\eta_1-1))} & G_{ij}^{(211\eta_1)} + G_{ij}^{(2221)} & G_{ij}^{(2222)} & \cdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{array} \right.
$$

$$
\left. \begin{array}{ccccc}
\ddots & \vdots & \vdots & \vdots & \vdots \\
\cdots & G_{ij}^{(N_e(N_e-1)1\eta_{N_e-1})} + G_{ij}^{(N_e N_e 11)} + G_{ij}^{((N_e-1)N_e\eta_{N_e-1}1)} + G_{ij}^{((N_e-1)(N_e-1)\eta_{N_e-1}\eta_{N_e-1})} & G_{ij}^{(N_e N_e 12)} + G_{ij}^{((N_e-1)N_e\eta_{N_e-1}2)} & \cdots & G_{ij}^{(N_e N_e 1\eta_{N_e})} + G_{ij}^{((N_e-1)N_e\eta_{N_e-1}\eta_{N_e})} \\
\cdots & G_{ij}^{(N_e(N_e-1)2\eta_{N_e-1})} + G_{ij}^{(N_e N_e 21)} & G_{ij}^{(N_e N_e 22)} & \cdots & G_{ij}^{(N_e N_e 2\eta_{N_e})} \\
\cdots & \vdots & \vdots & \ddots & \vdots \\
\cdots & G_{ij}^{(N_e N_e \eta_{N_e} 1)} + G_{ij}^{(N_e(N_e-1)\eta_{N_e}\eta_{N_e-1})} & G_{ij}^{(N_e N_e \eta_{N_e} 2)} & \cdots & G_{ij}^{(N_e N_e \eta_{N_e}\eta_{N_e})}
\end{array} \right\} \quad (27)
$$

.

The vectors $(\delta u_i)$, $(u_i)$ and $(T_{j2})$ are respectively defined by equations (26), (27) and (28):

$$< \delta u_i >= \left(\delta u_i^{11}, \quad \cdots, \quad \delta u_i^{1(\eta_1-1)}, \quad \delta u_i^{1\eta_1} + \delta u_i^{21}, \quad \delta u_i^{22}, \quad \cdots, \quad \delta u_i^{N_e\eta_{N_e}}\right), \tag{28}$$

$$\{u_i\} = \left(u_i^{11}, \quad \cdots, \quad u_i^{1(\eta_1-1)}, \quad u_i^{1\eta_1} + u_i^{21}, \quad u_i^{22}, \quad \cdots, \quad u_i^{N_e\eta_{N_e}}\right)^T \tag{29}$$

$$\{T_{j2}\} = \left(T_{j2}^{11}, \quad \cdots, \quad T_{j2}^{1(\eta_1-1)}, \quad T_{j2}^{1\eta_1} + T_{j2}^{21}, \quad T_{j2}^{22}, \quad \cdots, \quad T_{j2}^{N_e\eta_{N_e}}\right)^T \tag{30}$$

.

Also, whatever $< \delta u_i >$, we can write

$$\{T_{j2}\} = (X_{ij})\,\{u_i\}\,. \tag{31}$$

where $(X_{ij}) = (G_{ij})^{-1}(\Psi_{ij})$.

Thus, we can put the stress tensor $T$ in the right part of the equation (1) and found the variational unknown. So equation (1) can be written here in a global manner:

$$< \delta u > \left[K - \omega^2\right]\{u\} =< \delta u > [X]\,\{u\}\,. \tag{32}$$

# References


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